Distinguishing Codes from Noise

Motivation

- Communication: synchronization and coding
- Synchronization: mostly done by training:

\[
\begin{array}{cccc}
 c_1 & c_2 & \ldots & c_n \\
 1 & 1 & \ldots & 1 \\
 \end{array}
\]

capacity achieving code

\[(1 - \frac{R}{2})n\] symbols \(\frac{R}{2}n\) symbols

Is that always good? Can we do better?

Distinguishing Codes from Noise

Asynchronous channel

\[\begin{align*}
W_{Y|X} &: \text{ input symbol to model that nothing is sent.} \\
X &\in \{x_1, x_2, \ldots, x_i\} \\
Y &\in \{y_1, y_2, \ldots, y_j\} \\
\end{align*}\]

\[c(1) \quad c(2) \quad \ldots \quad c(n)\]

Slotted simplification

Communicate in pre-defined timeslots:

\[\begin{align*}
\varepsilon^n &\quad \varepsilon^n &\quad c(i) &\quad \varepsilon^n &\quad c(j) &\quad \varepsilon^n \\
\end{align*}\]

detect & locate \(\xrightarrow{\text{slotted}}\) detection only

Mathematical Setup

For a channel code \(C = \{X^n(k)\}\) with rate \(R\), we have the following hypothesis testing problem:

\[
\begin{align*}
H_0 : & \quad Y_i \overset{i.i.d.}{\sim} W(\cdot|\cdot) \quad i = 1, 2, \ldots, n \\
H_1 : & \quad Y^n \sim W(\cdot|X^n(k)) \quad k \in \{1, 2, \ldots, M\} \\
\end{align*}
\]

Define

\[
\begin{align*}
P_m &\triangleq P\{|H_1 \rightarrow H_0\} = \exp(-nE_m) \\
P_f &\triangleq P\{|H_0 \rightarrow H_1\} = \exp(-nE_f) \\
\end{align*}
\]

Analysis objectives

Characterize the \(E_m - E_f\) trade-off at rate \(R\).

Special case: \(E_m = 0\)

Optimal \(E_f(R)\) when \(E_m = 0\)

Given \(E_m = 0\) and hence \(P_m \rightarrow 0\),

\[
E_f(R) = \max_{P_X(I(P_X; W) = R)} D(P_Y \| Q_*)
\]

- i.i.d. codebook with distribution \(P_X\)
- noise output distribution \(Q_* = W(\cdot|\cdot)\)
- use rate-achieving i.i.d. codebook rather than capacity-achieving codebook.

BSC Example

For a binary symmetric channel (BSC) with \(W(\cdot|\cdot) = \text{Bern}(u)\) \((u \leq 0.5)\), we can achieve

\[
E_f = D(\text{Bern}(s^*) \mid \text{Bern}(u))
\]

c.w. output dist. noise output dist.

where \(H_b(s^*) - H_b(\varepsilon) = R\).

General case: \(E_m \geq 0\)

Achievable \(E_f(R)\) when \(E_m \geq 0\)

Given \(P_m \leq \exp(-nE_m)\),

\[
E_f(R, E_m) = \max_{P_X(I(P_X; W) \geq R)} \min_{V : D(V \mid W|P_X) \leq E_m}
\]

\[
\left[ D(Q_Y \mid Q_*) + \{I(P_X, V) - R\} \right]^+
\]

- achieved by constant composition codebook with maximizing distribution \(P_X^*\).
- i.i.d. codebook is suboptimal in general.
- non-trivial converse for DMC is unknown.

Comparison with i.i.d. codebook and training for BSC channel

\[
\begin{align*}
\text{Figure:} & \quad E_f(R) \quad \text{for BSC with } \varepsilon = 0.01 \\
\text{(a) } R = 0.405 & \quad \text{(b) } R = 0.708 \\
\end{align*}
\]

\[
\text{Figure:} & \quad \text{Performance comparison between constant composition codebook, i.i.d. codebook, and training for BSC with } \varepsilon = 0.05 \text{ and } u = 0.5. \\
\end{align*}
\]

\[
\text{✓ Again, large gain over training at high rate.}
\]

Extensions & Connections

- AWGN channel, unequal error protection (UEP), …

Conclusion

For certain communication scenarios, designing codes for both detection and information transmission jointly achieves significantly larger detection error exponents than the traditional separate sync–coding approach.