Scalar Quantization with Noisy Partitions and its Application to Flash ADC Design

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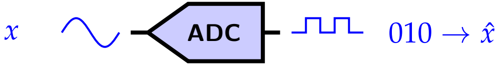
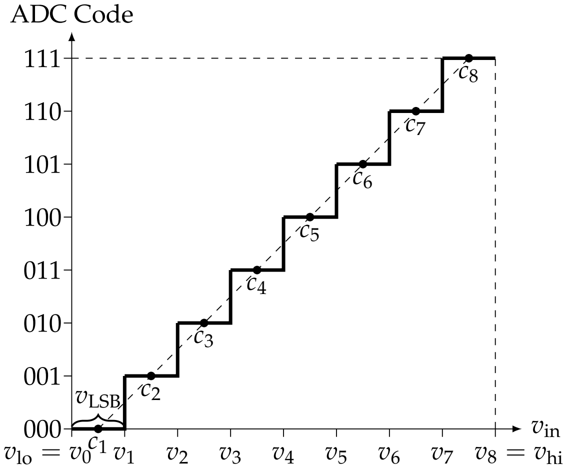
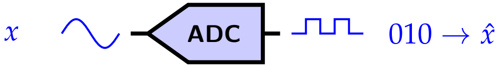
Outline

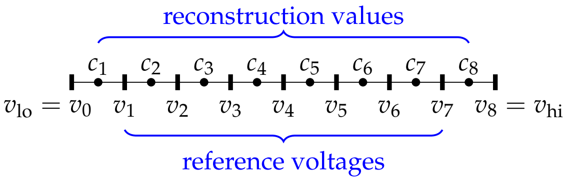
Analog-to-Digital Converter (ADC)





- 2^b reconstruction values
- $n = 2^b - 1$ reference voltages





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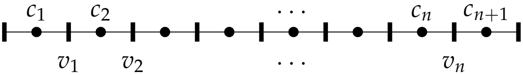
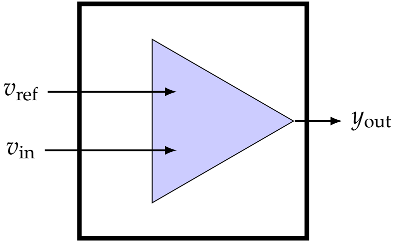
ADC and its key building block: comparator

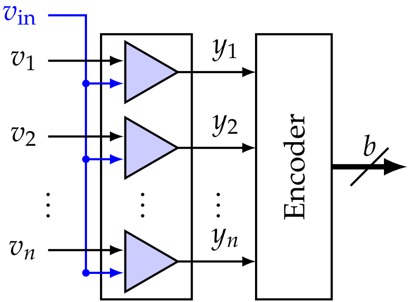


Comparator



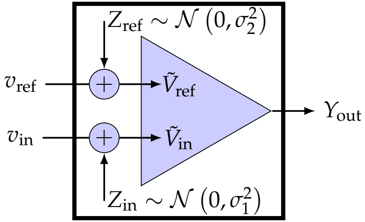
$$y_{\text{out}} = \begin{cases} 1 & v_{\text{in}} > v_{\text{ref}} \\ 0 & v_{\text{in}} \leq v_{\text{ref}} \end{cases}$$

The Flash ADC architecture



$$n = 2^b - 1$$

The imprecise comparator due to process variation



Z_{in} and Z_{ref}:

- offsets due to process variation
- variation $\nearrow$ as comparator size $\searrow$
- independent, zero-mean Gaussian distributed [Kinget 2005, Nuzzo 2008]

Note:

- fixed after fabrication
- randomness: over a collection of comparators
- aggregate variation:

$$Z = Z_{\text{ref}} - Z_{\text{in}} \sim \mathsf{N}\left(0, \sigma^2\right)$$

A call for mathematical framework

Existing theoretical error analysis (e.g., [Lundin 2005])

- assumes small process variation
- does not attempt to change the design

ADC design with imprecise comparators

Practice

- ADC with redundancy [Flynn *et al.*, 2003]
- ADC with redundancy, calibration and reconfiguration [Daly *et al.*, 2008]

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Practice
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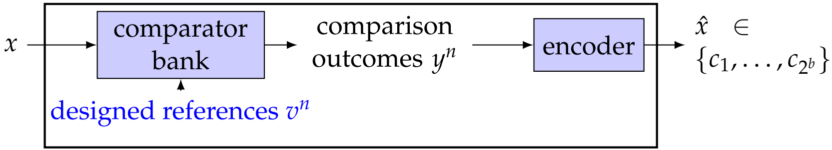
Theory
- Little prior work
- Related: scalar quantizer with random thresholds for uniform input [Goyal 2011]

b-bit ADC

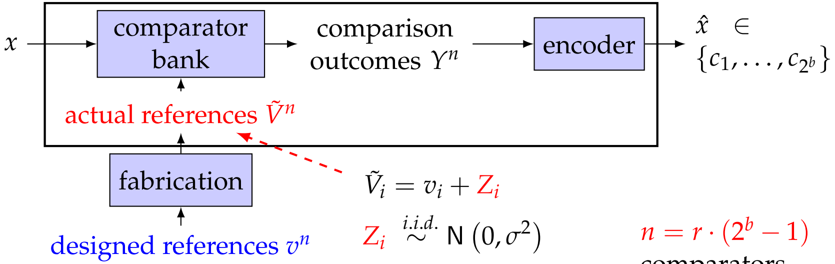
classical



$n = 2^b - 1$ comparators

b-bit ADC

with redundancy



$\tilde{V}_i = v_i + Z_i$

$Z_i \overset{i.i.d.}{\sim} \mathsf{N}\left(0, \sigma^2\right)$

$i = 1, 2, \ldots, n$

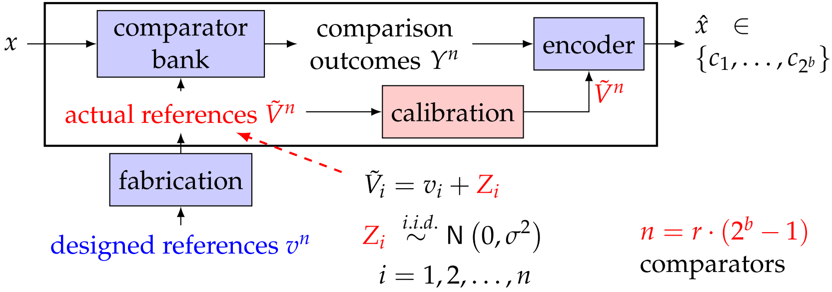
$n = r \cdot (2^b - 1)$ comparators

- "references" = "partition points"
- r: redundancy factor

System model: Scalar Quantization with Noisy Partition Points

b-bit ADC

with redundancy and calibration



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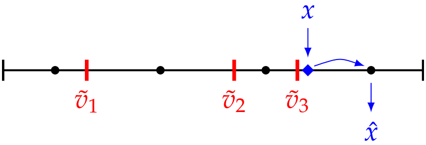
Performance measures of ADC



error function

$$e(x) = x - \hat{x}$$

mean-square error

$$\mathrm{MSE} = \mathbb{E}_{X,\tilde{V}^n}\left[e(X)^2\right]$$

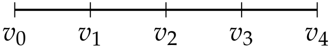
v^n — $\tilde{V}^n$ → analyze MSE

→ Given input distribution f_X, how to design optimal $v_1, v_2, \ldots, v_n$?

→ Is scaling down the size of comparators actually beneficial?

Challenge: randomness in partition points

e.g., design:



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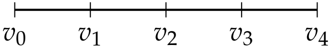
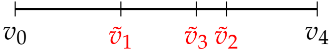


After fabrication:



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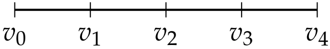

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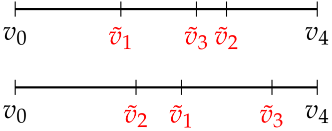
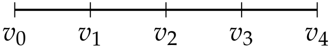
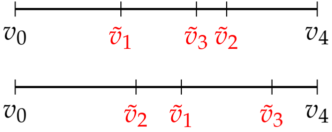


or …

e.g., design:



After fabrication:



or ...

Observations

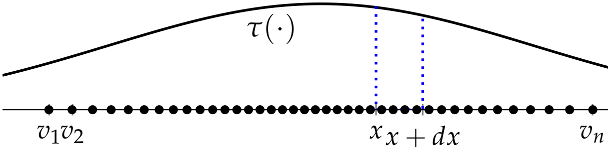
- Ordering may change $\rightarrow$ order statistics
- Random interval sizes $\rightarrow$?

interval sizes $\longleftrightarrow$ density of partition points $\longleftrightarrow$ high resolution approximation

Assume $n \to \infty$

- Represent v^n by point density functions $\tau(x)$

$$\tau(x)\,dx \approx \frac{\text{number of } v^n \text{ in } [x, x+dx]}{n}$$



High resolution approximation

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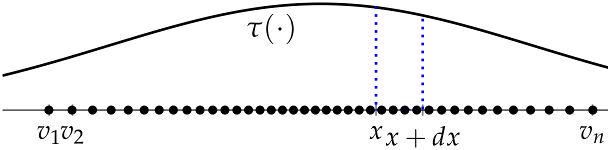
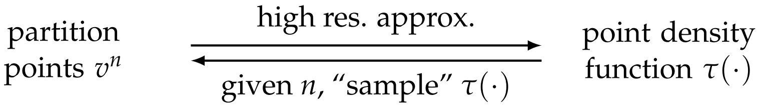


- $\tilde{V}^n$: point density functions $\lambda(x)$

$$\lambda(x)\,dx \approx \frac{\mathbb{E}\left[\text{number of } \tilde{V}^n \text{ in } [x, x+dx]\right]}{n}$$

- Point density function simplifies analysis!

Point density function guides partition point design



Examples

- $\tau \sim \mathsf{Unif}\left([-1,1]\right)$
- v^n: n-point evenly-spaced grid on [-1, 1]

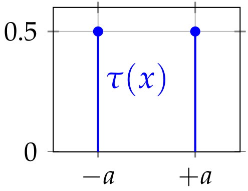
Point density function guides partition point design

partition points v^n —— high res. approx. ——→ point density function $\tau(\cdot)$

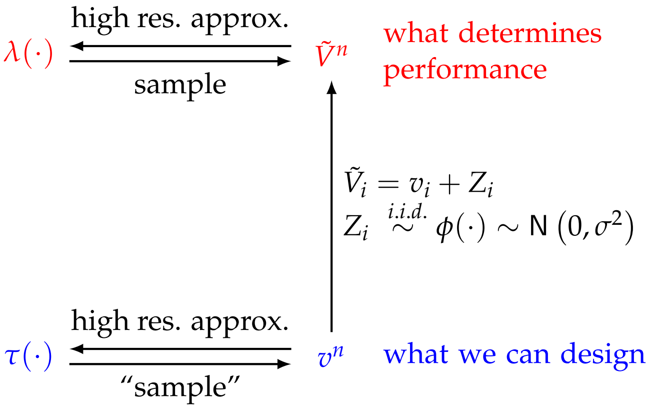
←—— given n, "sample" $\tau(\cdot)$ ——

Examples

- $\tau(x) = 0.5 \cdot \delta(x - a) + 0.5 \cdot \delta(x + a)$
- v^n:
  - $n/2$ points at $+a$
  - $n/2$ points at $-a$



With process variation, fabricated references matters



- Performance characterization in $\lambda(\cdot)$
- Want to find the optimal $\tau(\cdot)$

With process variation, fabricated references matters

$\lambda(\cdot)$ ⟵ high res. approx. / sample ⟶ $\tilde{V}^n$ — what determines performance

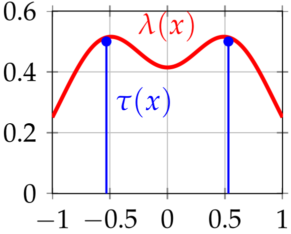
$\lambda(x) = (\tau * \phi)(x)$

($*$: convolution)

$\tilde{V}_i = v_i + Z_i$

$Z_i \overset{i.i.d.}{\sim} \phi(\cdot) \sim \mathsf{N}\left(0, \sigma^2\right)$

$\tau(\cdot)$ ⟵ high res. approx. / “sample” ⟶ v^n — what we can design



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Process variation increases MSE 6-fold

Input $X \sim f_X(\cdot)$,

$$\text{MSE} = \mathbb{E}_X\left[e(X)^2\right]$$

classical case [Bennett 1948, Panter & Dite 1951]

$$\text{MSE} \simeq \frac{1}{12n^2} \int \frac{f_X(x)}{\lambda^2(x)}\, dx \qquad \lambda = \tau$$

with process variations

$$\text{MSE} \simeq \frac{1}{2n^2} \int \frac{f_X(x)}{\lambda^2(x)}\, dx \qquad \lambda = \tau * \phi$$

Why 6 times?

deterministic grid vs. random division of an interval (a topic in order statistics)

Optimal τ

- a necessary and sufficient condition

Optimal partition point density

Key function:

$$R(\tau) = \int f_X(x)(\tau * \phi)^{-2}(x)\, dx$$

Theorem

τ minimizes $R(\tau)$ if and only if

$$\sup_{x \in \mathcal{A}} \left[\frac{f_X}{(\tau * \phi)^3} * \phi \right](x) \leq \left\langle f_X, \frac{1}{(\tau * \phi)^2} \right\rangle .$$

In particular, if there exists τ^* such that

$$\tau^* * \phi \propto f_X^{1/3},$$

then τ^* minimizes $R(\tau)$ and

$$R(\tau^*) = \left(\int f_X^{1/3}(x) d\, x \right)^3 .$$

MSE-optimal designs can be quite different
Gaussian input distribution

Complete characterization of optimal τ

When

$$f_X \sim \mathsf{N}\left(0, \sigma_X^2\right),$$

then

$$\tau^* \sim \begin{cases} \mathsf{N}\left(0, 3\sigma_X^2 - \sigma^2\right) & \text{when } 3\sigma_X^2 > \sigma^2 \\ \delta(x) & \text{when } 3\sigma_X^2 \leq \sigma^2 \end{cases},$$

and

$$R(\tau^*) = \begin{cases} 6\sqrt{3}\pi\sigma_X^2 & \text{when } 3\sigma_X^2 > \sigma^2 \\ 2\pi\sigma^3 / \sqrt{\sigma^2 - 2\sigma_X^2} & \text{when } 3\sigma_X^2 \leq \sigma^2 \end{cases}.$$

MSE-optimal designs can be quite different

Uniform input distribution

$$f_X \sim \mathsf{Unif}\left([-1,1]\right)$$

$\sigma_0 \approx 0.7228$

$\sigma < \sigma_0$

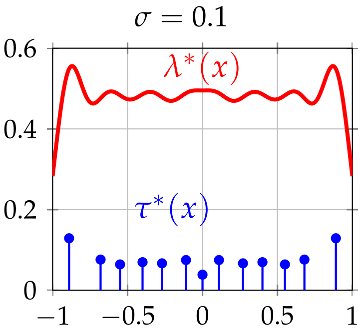
iterative optimization

$\Rightarrow$ locally optimal $\tau^*(x)$

$\sigma \geq \sigma_0$

the necessary and sufficient condition

$\Rightarrow \tau^*(x) = \delta(x)$



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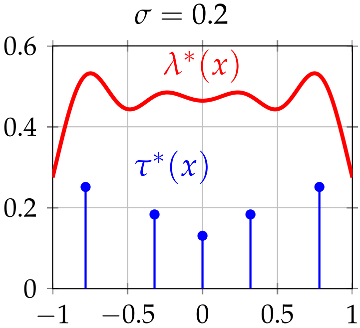

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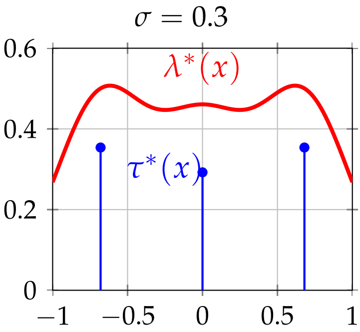

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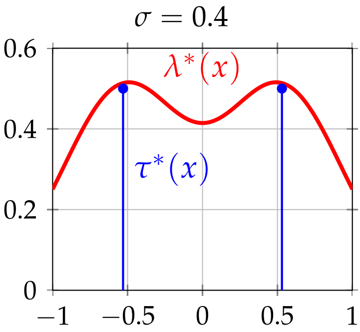

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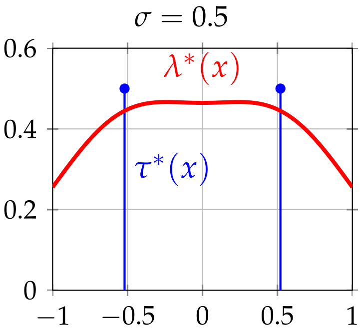

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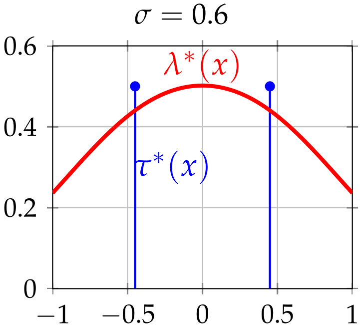

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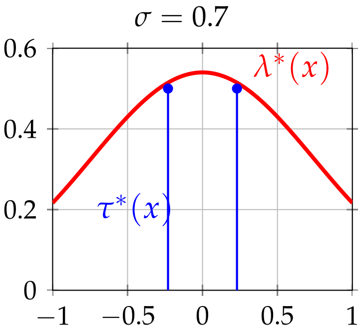

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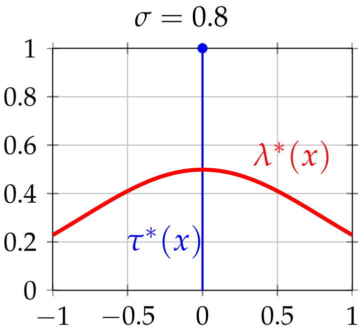
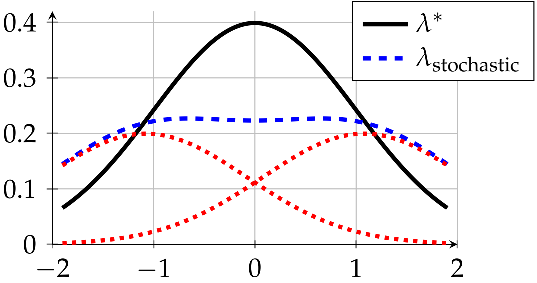


ADC design implications

Stochastic ADC [Weaver 2010] is suboptimal

- Uniform input distribution over [-1, 1], $\sigma = 1.0$



Weaver:
$\tau(x) = \delta(x - 1.078\sigma) + \delta(x + 1.078\sigma)$

- Flatter $\lambda(x)$, but larger MSE
- Many points out of input range

Optimal: $\tau(x) = \delta(x)$

- Minimum MSE

$$\text{MSE}_{\text{stochastic}} / \text{MSE}^* \approx 2.15!$$

Scaling down the size of comparators is beneficial

For circuit fabrication [Kinget 2005, Nuzzo 2008],

$$\text{process variation} \qquad \sigma^2 \propto \frac{1}{\text{component area}}$$

Given a fixed silicon area,

$$\text{\# components} \qquad n \propto \frac{1}{\text{component area}}$$

Uniform input distribution, when $\sigma \geq \sigma_0$,

$$\text{MSE} \approx 2\pi\sigma^2/n^2 \quad \xRightarrow{\sigma^2 \propto n} \quad \text{MSE} = \Theta\left(1/n\right)$$

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Building an ADC with more smaller but less precise comparators improves accuracy!

Recap

- Scalar quantization with noisy partitions
- High resolution analysis of MSE
- Optimal partition point designs difference from the classical case

Work in progress

- More error metrics: maximum quantization error, …
- Partial-calibration or no-calibration
- Take power consumption of ADC into account