A Rate-Distortion Theory for Permutation Spaces

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Codes in permutations

Error correction in permutations

- Codes with hamming distance: [I. Blake *et al.*, 1979]
- Codes with Chebyshev distance: [T. Klve *et al.*, 2010], [A. Barg and A. Mazumdar, 2010]
  - Application: rank modulation for flash memory

Lossy compression of permutations

- Largely left unattended
- Lossless compression of permutations for efficient rank query and selection: [J. Barbay and G. Navarro, 2009 & 2011], [J. Barbay *et al.*, 2012]

Given a group of elements with distinct values:

Comparison-based sorting: search for the true permutation by pairwise comparisons.

Algorithm 101: exact sorting

- To specify a permutation, need $\log_2 n! \approx \Theta(n \log n)$ bits.
- Each comparison: at most 1 bit of information
  $\Rightarrow$ need $\Omega(n \log n)$ comparisons

Given a group of elements with distinct values:

Comparison-based approximate sorting: search for the true permutation subject to certain distortion by pairwise comparisons.

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  $\Rightarrow$ need $\Omega(n \log n)$ comparisons

Approximate sorting

- How many comparisons do we need for sorting with distortion D?
- How many bits do we need for specifying a permutation with distortion D?

Permutation and (approximate) sorting

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Comparison-based approximate sorting: search for the true permutation subject to certain distortion by pairwise comparisons.

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Rate-distortion theory!

Permutation space

- $\mathcal{S}_n$: the set of $n!$ permutations
- d: distance measure

(n, D_n) source code

- $\mathcal{C}_n \subset \mathcal{S}_n$
- for any $\sigma \in \mathcal{S}_n$, there exists $\pi \in \mathcal{C}_n$ that
$$d(\pi, \sigma) \leq D_n.$$

Rate-distortion function

Let $A(n, D_n)$ be the minimum size of the (n, D_n) source codes with distortion D_n. The minimal rate for distortion D_n is

$$R(D_n) \triangleq \frac{\log A(n, D_n)}{\log n!},$$

and the rate-distortion function is

$$R(D) \triangleq \lim_{n \to \infty} R(D_n).$$

Vector representations

- the permutation vector σ
- the inverse permutation vector σ^{-1}: $\sigma \circ \sigma^{-1} = e = [1, 2, \dots, n]$
- the inversion vector of the permutation $\mathbf{x}_\sigma$

Distances between vectors

- Kendall tau distance
- ℓ_p distances, $p = 1, 2, \dots, \infty$

In this work

Two specific permutation spaces:

- Kendall tau distance of the permutation vectors
- ℓ_1 distance of the inversion vectors

- The *Kendall tau distance* $d_\tau(\sigma_1, \sigma_2)$:
  the minimum number of swaps of adjacent elements required to change σ_1 into σ_2.

Example

- $\sigma_1 = [1, 5, 4, 2, 3]$ and $\sigma_2 = [3, 4, 5, 1, 2]$
- $d_\tau\left(\sigma_1, \sigma_2\right) = ?$

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$$\sigma_1 = [1, 5, 4, \underline{2, 3}] \to [1, \underline{5, 4}, 3, 2]$$

Distance measure of permutations

Kendall tau distance

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$$\sigma_1 = [1, 5, 4, \underline{2, 3}] \rightarrow [1, \underline{5, 4}, 3, 2] \rightarrow [\underline{1, 4}, 5, 3, 2] \rightarrow [4, \underline{1, 5}, 3, 2]$$

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Example

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$$\rightarrow [4, 5, \underline{1, 3}, 2] \rightarrow [4, \underline{5, 3}, 1, 2] \rightarrow [\underline{4, 3}, 5, 1, 2]$$

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Kendall tau distance

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$$\begin{aligned}\sigma_1 = [1, 5, 4, \underline{2, 3}] &\rightarrow [1, \underline{5, 4}, 3, 2] \rightarrow [\underline{1, 4}, 5, 3, 2] \rightarrow [4, \underline{1, 5}, 3, 2] \\ &\rightarrow [4, 5, \underline{1, 3}, 2] \rightarrow [4, \underline{5, 3}, 1, 2] \rightarrow [\underline{4, 3}, 5, 1, 2] \\ &\rightarrow [3, 4, 5, 1, 2]\end{aligned}$$

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  the minimum number of swaps of adjacent elements required to change σ_1 into σ_2.

Example

- $\sigma_1 = [1, 5, 4, 2, 3]$ and $\sigma_2 = [3, 4, 5, 1, 2]$
- $d_\tau(\sigma_1, \sigma_2) = 7$

$$\sigma_1 = [1, 5, 4, \underline{2, 3}] \to [1, \underline{5, 4}, 3, 2] \to [\underline{1, 4}, 5, 3, 2] \to [4, \underline{1, 5}, 3, 2]$$
$$\to [4, 5, \underline{1, 3}, 2] \to [4, \underline{5, 3}, 1, 2] \to [\underline{4, 3}, 5, 1, 2]$$
$$\to [3, 4, 5, 1, 2] = \sigma_2$$

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Properties

- upper bounded by $\binom{n}{2}$
- $d_\tau(\sigma, e)$ = number of swaps in bubble sort

Inversion

- An *inversion* in a permutation σ: a pair $(\sigma(i), \sigma(j))$ such that $i < j$ and $\sigma(i) > \sigma(j)$.
  - Inversions in $\sigma_1 = [1, 5, 4, 2, 3]$: $(5, 4), (5, 2), (5, 3), (4, 2), (4, 3)$
  - Inversions in $\sigma_2 = [3, 4, 5, 1, 2]$: $(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)$

Inversion vector $\mathbf{x}_\sigma \in [0 : 1] \times [0 : 2] \times \cdots \times [0 : n-1]$

$$\mathbf{x}_\sigma(i) = \text{the number of inversions in } \sigma \text{ in which } i+1 \text{ is the first element}$$
$$i = 1, 2, \ldots, n-1.$$

Examples

$$\sigma_1 = [1, 5, 4, 2, 3] \Rightarrow \mathbf{x}_{\sigma_1} = [0, 0, 2, 3]$$
$$\sigma_2 = [3, 4, 5, 1, 2] \Rightarrow \mathbf{x}_{\sigma_2} = [0, 2, 2, 2]$$
$$d_{\mathbf{x},\ell_1}(\sigma_1, \sigma_2) = d_{\ell_1}([0, 0, 2, 3], [0, 2, 2, 2]) = 3$$

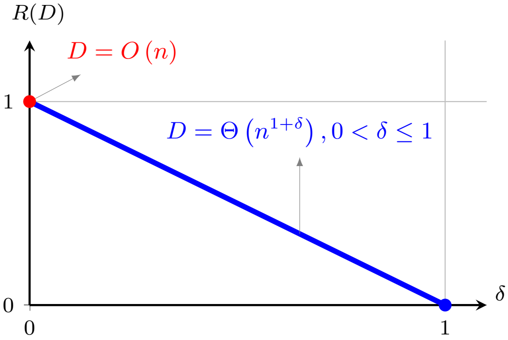
- Inversion vector: a common measure of sortedness
- $d_{\mathbf{x},\ell_1}(e, \sigma)$: evaluation metric for ranking system

Main result

Theorem (Rate distortion function)

In both permutation spaces $\mathcal{X}(\mathcal{S}_n, d_\tau)$ *and* $\mathcal{X}(\mathcal{S}_n, d_{\mathbf{x},\ell_1})$,

$$R(D) = \begin{cases} 1 & \text{if } D = O(n) \\ 1-\delta & \text{if } D = \Theta\left(n^{1+\delta}\right), \quad 0 < \delta \leq 1 \end{cases}.$$



Remarks

- Given two permutations σ_1 and σ_2, [A. Mazumdar *et al.*, 2013] shows

$$d_{\mathbf{x},\ell_1}(\sigma_1,\sigma_2) \leq d_\tau(\sigma_1,\sigma_2)$$

  - (n,D) code for $d_\tau(\cdot,\cdot) \Rightarrow (n,D)$ code for $d_{\mathbf{x},\ell_1}(\cdot,\cdot)$
  - No known non-trivial lower bound for $d_{\mathbf{x},\ell_1}(\sigma_1,\sigma_2)$ in terms of $d_\tau(\sigma_1,\sigma_2)$.

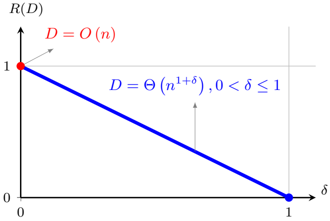
- In the moderate distortion regime, where

$$D = \Theta\left(n^{1+\delta}\right), \quad 0 \leq \delta < 1,$$

the rate for worst-case distortion is the same for average-case distortion with uniform distribution on $\mathcal{S}_n$.

Two end-points of the rate distortion function

$$D_n = O(n) \Rightarrow \quad R(D) = 1 \quad \text{(small distortion)}$$
$$D_n = \Omega(n^2) \Rightarrow \quad R(D) = 0 \quad \text{(large distortion).}$$



Higher order rates in the codebook size

$$r(D_n) \triangleq \log A(n, D_n) - R(D) \cdot \log n!$$

- $r(D_n) \leq 0$ when $R(D) = 1$
- $r(D_n) \geq 0$ when $R(D) = 0$
- Exact characterization is open: present upper and lower bounds.
  - upper bound: achievability
  - lower bound: converse

Kendall tau distance

In the permutation space $\mathcal{X}(\mathcal{S}_n, d_\tau)$, when $D = an^\delta, 0 < \delta \leq 1$,

$$\underline{r^{\mathrm{s}}_\tau(D)} \leq r(D) \leq \overline{r^{\mathrm{s}}_\tau(D)},$$

where

$$\underline{r^{\mathrm{s}}_\tau(D)} = \begin{cases} -a(1-\delta)n^\delta \log n + O\left(n^\delta\right), & 0 < \delta < 1 \\ -n\left[(1+a)\log(1+a) - a\log a\right] + o(n), & \delta = 1 \end{cases},$$

$$\overline{r^{\mathrm{s}}_\tau(D)} = \begin{cases} -n^\delta \frac{a\log 2}{2} + O(1), & 0 < a < 1 \\ -n^\delta \frac{\log \lfloor 2a \rfloor !}{\lfloor 2a \rfloor} + O(1), & a \geq 1 \end{cases}.$$

When $D = bn^2, 0 < b \leq 1/2$, $\underline{r^{\mathrm{l}}_\tau(D)} \leq r(D) \leq \overline{r^{\mathrm{l}}_\tau(D)}$, where

$$\underline{r^{\mathrm{l}}_\tau(D)} = \max\left\{0, n\log 1/\left(2be^2\right)\right\},$$

$$\overline{r^{\mathrm{l}}_\tau(D)} = n\log\lceil 1/(2b)\rceil + O(\log n).$$

ℓ_1 distance of inversion vectors

In the permutation space $\mathcal{X}\left(\mathcal{S}_n, d_{\mathbf{x},\ell_1}\right)$, when $D = an^\delta, 0 < \delta \leq 1$,

$$\underline{r^{\mathrm{s}}_{\mathbf{x},\ell_1}(D)} \leq r(D) \leq \overline{r^{\mathrm{s}}_{\mathbf{x},\ell_1}(D)},$$

where $\underline{r^{\mathrm{s}}_{\mathbf{x},\ell_1}(D)} = \underline{r^{\mathrm{s}}_{\tau}(D)} - n^\delta \log 2$ and

$$\overline{r_{\mathbf{x},\ell_1}(D)} = \begin{cases} -\left\lfloor n^\delta \right\rfloor \log(2a-1) & a > 1 \\ -\left\lceil an^\delta \right\rceil \log 3 & 0 < a \leq 1 \end{cases}.$$

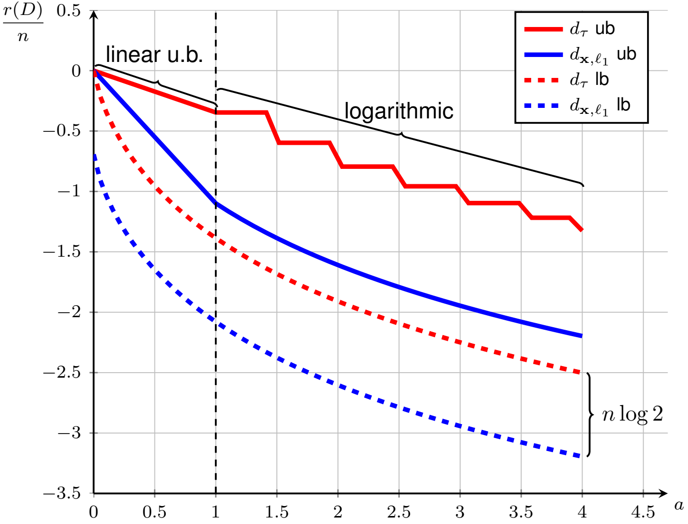
When $D = bn^2, 0 < b \leq 1/2$,

$$\underline{r^{\mathrm{l}}_{\mathbf{x},\ell_1}(D)} \leq r(D) \leq \overline{r^{\mathrm{l}}_{\mathbf{x},\ell_1}(D)},$$

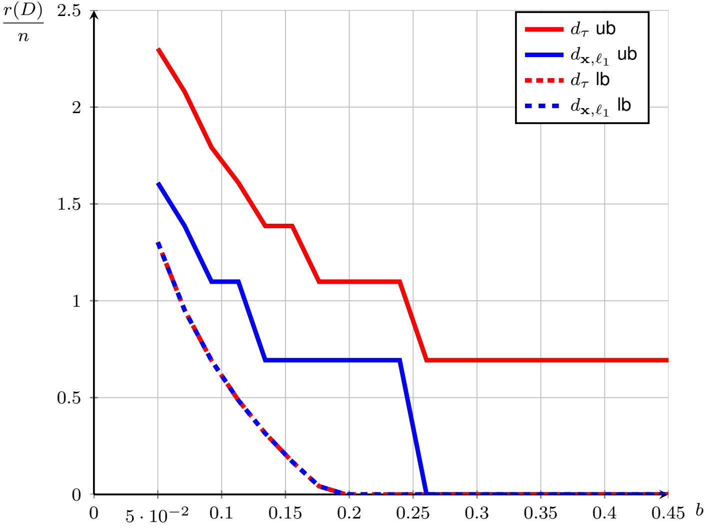
where $\underline{r^{\mathrm{l}}_{\mathbf{x},\ell_1}(D)} = \underline{r^{\mathrm{l}}_{\tau}(D)}$ and $\overline{r^{\mathrm{l}}_{\mathbf{x},\ell_1}(D)} = n \log \left\lceil 1/(4b) \right\rceil + O\left(1\right).$

Higher order rates

Small distortion region: $D = an$



Large distortion region: $D = bn^2$



Achievability

Achievability for Kendall tau distance

Sorting subsequences

Quantization by sorting subsequences

Given $\sigma \in \mathcal{S}_n$, we quantize it to π by sorting k subsequence with length m

$\sigma_1, \sigma_2, \ldots, \sigma_m$	$\sigma_{m+1}, \ldots, \sigma_{2m}$	$\cdots$	$\sigma_{(k-1)m+1}, \ldots, \sigma_{km}$	$\sigma_{km+1}, \ldots, \sigma_n$
sort	sort		sort	keep

- **Codebook size**

$$|\mathcal{C}(k, m, n)| = n!/\left(m!^k\right)$$

- **Maximal distortion**

$$D(k, m) \leq km^2/2$$

To achieve moderate distortion $D = \Theta\left(n^{1+\delta}\right), 0 \leq \delta < 1$

$$\begin{cases} m \approx 2D/n = \Theta\left(n^{\delta}\right) \\ k \approx n/m = \Theta\left(n^{1-\delta}\right) \end{cases} \Rightarrow |\mathcal{C}(k, m, n)| \approx n \log n - k \cdot m \log m = (1-\delta) n \log n$$

Component-wise scalar quantization

- Quantize the k-th coordinate uniformly by m_k points ($k = 2, 3, \ldots, n$)
- Product structure of the space: $[0:1] \times [0:2] \times \cdots \times [0:n-1]$

- Codebook size:

$$M_n = \prod_{k=2}^{n} m_k$$

- Maximal distortion

$$D_n = \sum_{k=2}^{n} D_k$$

$$D_k = \lceil (k/m_k - 1)/2 \rceil$$

To achieve moderate distortion $D = \Theta\left(n^{1+\delta}\right), 0 \le \delta < 1$

$$m_k \approx \frac{n^2}{2D} = \Theta\left(n^{1-\delta}\right) \quad \Rightarrow \begin{cases} D_k & \approx \frac{kD}{n^2} = k\Theta\left(D^{1-\delta}\right) \\ M_n & \approx (1-\delta) n \log n \end{cases}.$$

Converse

D-balls $B_d(\sigma, D)$

- Distance measure: $d(\cdot, \cdot)$
- Center: σ
- Radius: D
- Maximum size: $N_d(D)$.

$n!$ divided by the upper bound of $N_d(D)$ provides converse results.

Key lemmas via combinatorial arguments

Kendall tau distance

For $0 \leq D \leq n$,

$$N_\tau (D) \leq \binom{n+D-1}{D}.$$

ℓ_1 distance of the inversion vectors

For $0 \leq D \leq n(n-1)/2$,

$$N_{\mathbf{x},\ell_1} (D) \leq 2^{\min\{n,D\}} \binom{n+D}{D}.$$

Recap

- Information theory provides the fundamental trade-off between complexity and accuracy in approximate sorting.
  - Can be generalized to other comparison-based algorithms.
- Achievability: sorting subsequences and quantizing coordinates
  - both support successive refinement
- Converse: geometry of the permutation spaces

Future directions

- Sharper bounds for higher order rates
- Other distance measures of interest
- Design of approximate sorting algorithms

Backup slides

Equivalent procedure in the inverse permutation domain

1. Construct a vector $a(\sigma)$ such that for $1 \leq i \leq k$,

$$a(i) = j \text{ if } \sigma^{-1}(i) \in [(j-1)m+1, jm], 1 \leq j \leq k.$$

   Then a contains exactly m values of integers j.
2. Form a permutation π' by replacing the length-m subsequence of a that corresponds to value j by vector $[(j-1)m+1, (j-1)m+2, \ldots, jm]$.

- The total number of inversions in σ is $I_n(\sigma)$.
  - $I_5(\sigma_1) = 5$, $I_5(\sigma_2) = 6$
- The number of permutations with k inversions:

$$K_n(k) \triangleq |\{\sigma \in \mathcal{S}_n : I_n(\sigma) = k\}|$$

Lemma

For $0 \leq D \leq n$,

$$N_{\tau}(D) \leq \binom{n+D-1}{D}.$$

Proof sketch.

Let the number of permutations in $\mathcal{S}_n$ with at most k inversions be $T_n(d) \triangleq \sum_{k=0}^{d} K_n(k)$. Then

$$N_{\tau}(D) = T_n(D),$$

By induction, $T_n(D) = K_{n+1}(D)$ when $D \leq n$. Then noting that for $k < n$, $K_n(k) = K_n(k-1) + K_{n-1}(k)$ ([Knuth 1998, Section 5.1.1]) and for any $n \geq 2$,

$$K_n(0) = 1, \quad K_n(1) = n-1, \quad K_n(2) = \binom{n}{2} - 1,$$

The proof can be completed by induction. □

Lemma

For $0 \leq D \leq n(n-1)/2$*,*

$$N_{\mathbf{x},\ell_1}(D) \leq 2^{\min\{n,D\}}\binom{n+D}{D}.$$

Proof sketch.

For any $\sigma \in \mathcal{S}_n$, let $\mathbf{x} = \mathbf{x}_\sigma \in \mathcal{G}_n$, then $|B_{\mathbf{x},\ell_1}(D)| = \sum_{r=0}^{D} |\{\mathbf{y} \in \mathcal{G}_n : d_{\ell_1}(\mathbf{x},\mathbf{y}) = r\}|$. Let $\mathbf{d} \triangleq |\mathbf{x}-\mathbf{y}|$, and $Q(n,r)$ be the number of integer solutions of the equation $z_1 + z_2 + \ldots + z_n = r$ with $z_i \geq 0, 0 \leq i \leq n$, then noting $|\{\mathbf{y} \in \mathcal{G}_n : d_{\ell_1}(\mathbf{x},\mathbf{y}) = r\}| \leq 2^{\min\{n,D\}}Q(n,r)$ and by upper bounding $Q(n,r)$ we complete the proof. □