On Reliability Functions for Single-Message Unequal Error Protection

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Classical channel coding setup

- All messages equally important
- Uniform protection
  - Average error:

$$P_e = \frac{1}{M} \sum_{m=1}^{M} P_e(m)$$

  - Maximal error:

$$P_e = \max_{1 \leq m \leq M} P_e(m)$$

- What if we have non-uniform protection?

Unequal error protection (UEP)

- Protect special messages different from regular messages.
- Multi-message UEP: exponentially many special messages
- Single-message UEP: one special message
  - Scope of this talk

Single-message UEP: three types of errors

- **miss**: decode the special codeword as a regular codeword,
- **false alarm**: decode a regular codeword as the special codeword, and
- **decoding error**: decode a regular codeword to another regular codeword.

Single-message unequal error protection

Motivation and applications

Communication with delay requirements

- Example: distributed control, media streaming
- Allowing non-block encoding schemes improves performance [B. D. Kudryashov, 1979]
- Special message as NACK signal [S. Draper and A. Sahai, 2006]
  - Achieves an error exponent much higher than the traditional channel coding error exponent
  - See also [A. Sahai and S. Draper, 2008]

Slotted asynchronous communication

- Channel output induced by noise $\approx$ channel output induced by a special codeword with repeated symbols [D. Wang, V. Chandar, S. Chung, and G. W. Wornell, 2011]
- UEP with constraints on the special codeword design

Single-message unequal error protection

Problem setup

- Channel $W : \mathcal{X} \to \mathcal{Y}$
- Message set $\mathcal{M}_{f_n} = \{0, 1, 2, \ldots, |\mathcal{M}_{f_n}|\}$
  - Message 0: the special message
- Encoder $f_n : \mathcal{M}_{f_n} \to \mathcal{X}^n$
- Decoder $g_n : \mathcal{Y}^n \to \mathcal{M}_{f_n}$.
  - Decoding region for regular codewords: $\mathcal{A}_n \triangleq \cup_{m \neq 0} g^{-1}(m)$
  - Decoding region for the special codeword: $\mathcal{B}_n \triangleq g_n^{-1}(m = 0)$

Given a codebook $\mathcal{C}^{(n)}$, performance metrics:

- Rate: R
- Miss probability: P_{m}
- False alarm probability: P_{f}
- Decoding error probability: P_{e}
- Error exponents: E_{m}, E_{f}, E_{d}

Single-message unequal error protection

Central question

Given $P_{\mathrm{m}}, P_{\mathrm{f}}, P_{\mathrm{e}} \to 0$,

- Allowing $E_{\mathrm{f}} = 0$, what is the maximum E_{m}?
  - miss reliability function $E_{\mathrm{m}}(R)$
- Allowing $E_{\mathrm{m}} = 0$, what is the maximum E_{f}?
  - false alarm reliability function $E_{\mathrm{f}}(R)$

First proposed and investigated by [S. Borade, B. Nakiboğlu, and L. Zheng, 2009]

Single-message unequal error protection

Previous work and our contributions

Maximize E_{m}

- BSC [1]
- DMC [2, 4]
- AWGN [3]
- Our contribution: a simpler converse proof for DMC

Maximize E_{f}

- Lower and upper bounds for DMC at $R = C$ [2]
- Our contribution: Extend the lower and upper bounds to all rates up to capacity for DMC

All results are obtained via a few generalizations of standard results in the method of types.

[1] Sahai and Draper, "The "hallucination" bound for the BSC," 2008

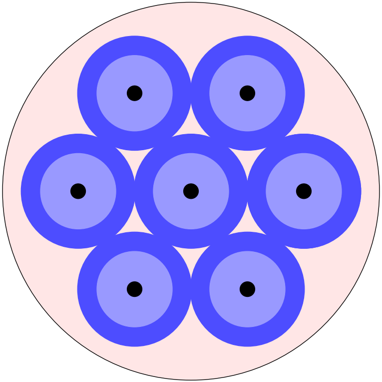
[2] Borade *et al.*, "Unequal error protection: An Information-Theoretic perspective," 2009

[3] Nazer *et al.*, "The AWGN red alert problem," 2011

[4] Gorantla *et al.*, "Bit-wise unequal error protection for variable length blockcodes with feedback," 2010

Reliability functions: some geometric intuitions

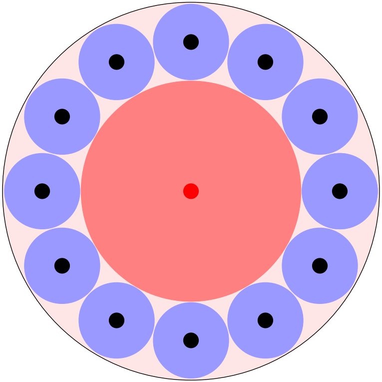
Decoding reliability function (channel coding error exponents)



- Maximize the “decoding shell” around each codeword.
- Non-overlapping

Reliability functions: some geometric intuitions

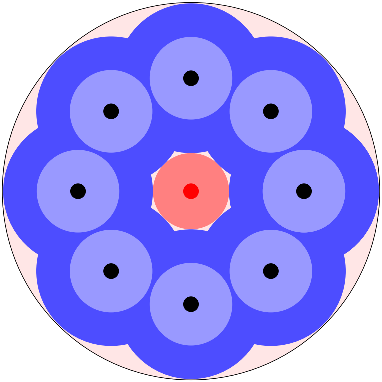
Miss reliability function



- $P_{\mathrm{e}} \to 0$
  - typical shell of each regular codeword cannot overlap each other
- $P_{\mathrm{f}} \to 0$
  - typical shell of each regular codeword cannot overlap with $\mathcal{B}_n$
- Maximize E_{m}
  - maximize the "detection shell" for the special codeword.

Reliability functions: some geometric intuitions

False alarm reliability function



- $P_{\mathrm{e}} \to 0$
  - typical shell of each regular codeword cannot overlap!
- $P_{\mathrm{m}} \to 0$
  - typical shell of the special codeword cannot overlap with $\mathcal{A}_n$
- Maximize E_{f}:
  - a larger “detection shell” around each regular codeword
  - Cannot overalp with special codeword decoding region
  - But may overlap with each overlap

Technical results

Special codeword with repeated symbols

Given a special codeword $\mathbf{s} = b^n$, a DMC $(\mathcal{X}, \mathcal{Y}, W)$ has miss reliability function

$$E_{\mathrm{m}}^{\mathrm{rep}}(R) = \max_{P_X : I(P_X, W) = R} D\left(P_Y \,\|\, W_b\right).$$

where $P_Y \triangleq P_X W$, $W_b \triangleq W(\cdot \,|\, b)$.

- Achieved by a constant composition code with type P_X.

Special codeword with no constraints (general case)

A DMC $(\mathcal{X}, \mathcal{Y}, W)$ has miss reliability function

$$E_{\mathrm{m}}(R) = \max_{P_{X,S} : \mathbb{E}_{P_S}\left[I\left(P_{X|S}, W\right)\right] = R} \left[\sum_{b \in \mathcal{X}} P_S(b) D\left(P_{Y|S=b} \,\middle\|\, W_b\right)\right]$$

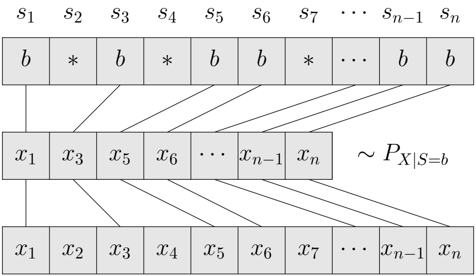
where $P_{Y|S=b} \triangleq P_{X|S=b} W$, $W_b \triangleq W(\cdot \,|\, b)$.

- Achieved by “time-sharing” constant composition codes.

Miss reliability function

Achievability

- Special codeword s with type P_S
- Index set $\mathcal{I}_b(s^n) \triangleq \{i : s_i = b\}$.
- Choose regular codewords that is constant composition with type $P_{X|S=b}$ on index set $\mathcal{I}_b(s^n)$.
- Constant composition w.r.t. s.



$$E_{\mathrm{m}}(R) = \max_{P_{X,S}:\mathbb{E}_{P_S}\left[I\left(P_{X|S},W\right)\right]=R} \left[\sum_{b\in\mathcal{X}} P_S(b) D\left(P_{Y|S=b} \,\middle\|\, W_b\right)\right]$$

rate constraint

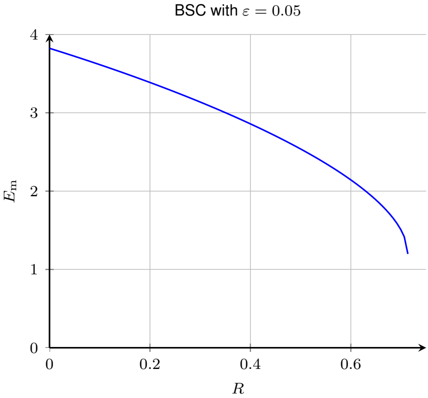
divergence contribution from each subsequence

Miss reliability function

Examples

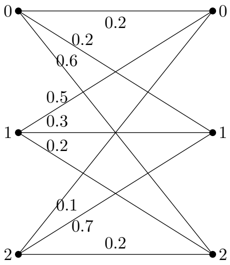
- $\varepsilon = 0.05$
- Special message with repeated symbol is optimal. [A. Sahai and S. Draper, 2008]

Binary symmetric channel

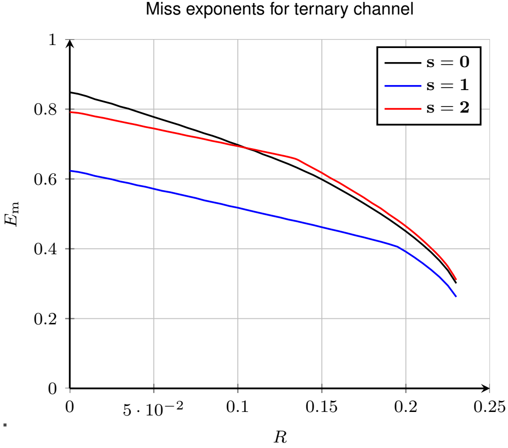


Miss reliability function
Examples

Ternary channel



- Special message with repeated symbols: suboptimal.
- Carathéodory's theorem ⇒ two symbols suffice.



Key lemmas:

- Given $\mathbf{s}$, every channel code contains a subcode that is constant composition w.r.t $\mathbf{s}$ with essentially the same rate.
  - polynomially many types w.r.t. $\mathbf{s}$
- For any $\mathbf{x}$ that is constant composition w.r.t $\mathbf{s}$, then its η-image size is roughly the size of its typical shell.
  - $\mathcal{B}$ is an η-image for $\mathbf{x}$: $W^n(\mathcal{B}|\mathbf{x}) \geq \eta$

Remarks:

- Constant composition code: proved in [I. Csiszár and J. Körner, 1981/2011]
- Generalized to constant composition code w.r.t $\mathbf{s}$ in this work.

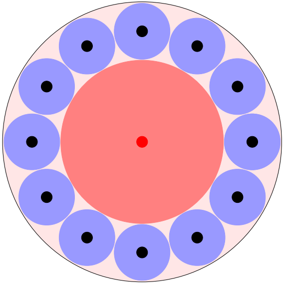
Miss reliability function

Converse proof

- Consider codes that are constant composition w.r.t. s
- $P_{\mathrm{e}} \to 0$: the decoding/detection region $\mathcal{A}_n$ for regular codewords cannot be too small

$$|\mathcal{A}_n| \dot{\geq} \exp\left[\sum_{b \in \mathcal{X}} n_b H\left(W|\hat{P}_{x_{\mathcal{I}_b}}\right)\right]$$

- Method of types: the probability that the special codeword s falls into $\mathcal{A}_n$ cannot be too small



Therefore,

$$P_{\mathrm{m}} = W^n\left(\mathcal{A}_n | s^n\right) \geq \cdots$$

False alarm reliability function

Theorem: bounds for the false alarm reliability function

The false alarm reliability function of an DMC $(\mathcal{X}, \mathcal{Y}, W)$ satisfies

$$\underline{E_{\mathrm{f}}}(R) \leq E_{\mathrm{f}}(R) \leq \overline{E_{\mathrm{f}}}(R),$$

where

$$\underline{E_{\mathrm{f}}}(R) \triangleq \max_{P_{X,S}:\mathbb{E}_{P_S}\left[I\left(P_{X\,|\,S},W\right)\right]\geq R} \left[\sum_{b\in\mathcal{X}} P_S(b) \cdot \min_{V:P_{X\,|\,S=b}V=W_b} \sum_{a\in\mathcal{X}} P_{X\,|\,S=b}(a) D\left(V_a \,\|\, W_a\right) \right],$$

$$\overline{E_{\mathrm{f}}}(R) \triangleq \max_{P_{X,S}:\mathbb{E}_{P_S}\left[I\left(P_{X\,|\,S},W\right)\right]\geq R} \sum_{a,b\in\mathcal{X}} P_{X,S}(a,b) D\left(W_b \,\|\, W_a\right).$$

False alarm reliability function

Corollary: bounds for the false alarm reliability function at capacity

Let the set of capacity-achieving input distributions be
$\Pi = \{P_X : I(P_X, W) = C\}$, then

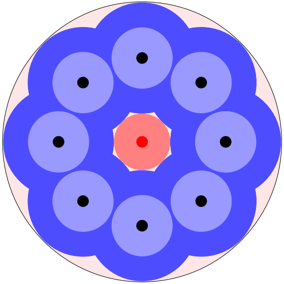
$$\underline{E_{\mathrm{f}}}(R = C) = \max_{P_X^* \in \Pi} \max_{b \in \mathcal{X}} \min_{V : P_X^* V = W_b} D\left(V \,\|\, W | P_X^*\right),$$

$$\overline{E_{\mathrm{f}}}(R = C) = \max_{P_X^* \in \Pi} \max_{b \in \mathcal{X}} \sum_{a \in \mathcal{X}} P_X^*(a) D\left(W_b \,\|\, W_a\right).$$

- Match the results in [S. Borade, B. Nakiboğlu, and L. Zheng, 2009].

False alarm reliability function

Special codeword with repeated symbols b



- Special codeword: b^n
-

$$\mathcal{V}_b \triangleq \{V : P_X V = W_b\},$$

$$\underline{E_{\mathrm{f}}^{\mathrm{rep}}}(R) \triangleq \max_{P_X : I(P_X, W) \geq R} \min_{V \in \mathcal{V}_b} D\left(V \,\|\, W | P_X\right),$$

$$\overline{E_{\mathrm{f}}^{\mathrm{rep}}}(R) \triangleq \min_{P_X : I(P_X, W) \geq R} D\left(W_b \,\|\, W | P_X\right).$$

- Shown in [D. Wang, V. Chandar, S. Chung, and G. W. Wornell, 2011]
- Then apply the time sharing argument for the unconstrained case.

Concluding remarks

- New proof and new results for single-message unequal error protection.
- Implications on optimal special message design.
- Useful building block for system design (streaming, synchronization, etc.)

Future directions

- Sharper inner and outer bounds for $E_{\mathrm{f}}(R)$
- Trade-off between $E_{\mathrm{m}}(R)$ and $E_{\mathrm{f}}(R)$
  - [D. Wang, V. Chandar, S. Chung, and G. W. Wornell, 2011] investigated the case of special codeword with repeated symbols.

Backup slides

Key lemmas for converse proof

Constant composition subcode w.r.t s

For any $\lambda > 0$, if a code (f_n, g_n) satisfies

$$|\mathcal{M}_{f_n}| \geq e^{n(R-\lambda)}$$

and $P_\mathrm{e} < \varepsilon$ for a given channel W, then for any given s^n with type P_S and $\mathcal{I}_b \triangleq \mathcal{I}_b(s^n)$, there exists a collection of types $\{P_b, b \in \mathcal{X}\}$ and a subcode $(\hat{f}_n, \hat{g}_n)$ with all regular codewords in the set

$$\hat{\mathcal{C}} \triangleq \left\{x^n : x_{\mathcal{I}_b} \in \mathcal{T}_{P_b}^{n_b}\right\},$$

where such that

$$\left|\mathcal{M}_{\hat{f}_n}\right| \geq \exp\left\{n\left[R - 2\lambda\right]\right\}$$

and

$$\sum_{b\in\mathcal{X}} P_S(b) I\left(P_b, W\right) \geq R - 3\lambda.$$

when n sufficiently large.

Key lemmas for converse proof

Generalized finite probability lemma

Given a sequence s^n with type P_S and $\mathcal{I}_b \triangleq \mathcal{I}_b(s^n)$, if a set $\mathcal{B} \subset \mathcal{Y}^n$ satisfies

$$W^n \left(\mathcal{B} \middle| x^n \right) \geq \eta,$$

then for any $\tau > 0$

$$|\mathcal{B}| \geq \exp \left\{ \left[\sum_{b \in \mathcal{X}} n_b H \left(W | \hat{P}_{x_{\mathcal{I}_b}} \right) - \tau \right] \right\}.$$

False alarm reliability function

Achievability for special codeword without constraints

- Represent each message m by a $|\mathcal{X}|$-tuple $(i_1, i_2, \ldots, i_{|\mathcal{X}|})$, where $i_b \in \{0, 1, 2, \ldots, |\mathcal{C}_b|\}, b \in \mathcal{X}$.
- Encoding function

$$f\left((i_1, i_2, \ldots, i_{|\mathcal{X}|})\right) = \begin{cases} x^n \text{ with } x_{\mathcal{I}_b} = f_b(i_b) & \text{when } i_b > 0 \;\forall\; b \in \mathcal{X} \\ s^n & \text{when } i_b = 0 \;\forall\; b \in \mathcal{X} \end{cases}$$

- Decoding function

$$g\left(y^n\right) = \left(\hat{i}_1, \hat{i}_2, \ldots, \hat{i}_{|\mathcal{X}|}\right) \text{ with } \hat{i}_b = g_b(y_{\mathcal{I}_b}) \;\forall\; b \in \mathcal{X}.$$

False alarm reliability function: achievability

Error analysis

Error events:

$$\mathcal{E}_{\rm e} = \{\text{any } g_b \text{ reports an error }\},$$
$$\mathcal{E}_{\rm m} = \left\{g_b(y_{\mathcal{I}_b}) \neq 0, \exists\, b \in \mathcal{X} \mid i_1 = i_2 = i_{|\mathcal{X}|} = 0\right\},$$
$$\mathcal{E}_{\rm f} = \left\{g_b(y_{\mathcal{I}_b}) = 0, \forall\, b \in \mathcal{X} \mid i_1 i_2 \cdots i_{|\mathcal{X}|} \neq 0\right\},$$

Error probabilities:

$$P_{\rm e}\left(\mathcal{C}^{(n)}\right) \leq \sum_{b \in \mathcal{X}} P_{\rm e}\left(C_b\right),$$
$$P_{\rm m}\left(\mathcal{C}^{(n)}\right) \leq \sum_{b \in \mathcal{X}} P_{\rm m}\left(C_b\right),$$
$$P_{\rm f}\left(\mathcal{C}^{(n)}\right) = \prod_{b \in \mathcal{X}} P_{\rm f}\left(C_b\right) \dot{\leq} \exp\left\{-n \sum_{b \in \mathcal{X}} P_S(b) \left[\min_{V \in \mathcal{V}_b} D\left(V \,\|\, W | P_b\right)\right]\right\}.$$